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ON SURROGATE DUALITY FOR ROBUST SEMI-INFINITE OPTIMIZATION PROBLEM

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Abstract. A semi-infinite optimization problem involving a quasi-convex objective function and infinitely many convex constraint functions with data uncertainty is considered. A surrogate duality theorem for the semi-infinite optimization problem is given under a closed and convex cone constraint qualification.

1. Introduction

Optimization problems in the face of data uncertainty have been treated by the worst case approach(the robust approach) or the stochastic approach. The worst case approach for optimization problems, which has emerged as a powerful deterministic approach for studying optimization problems with data uncertainty, associates an uncertain optimization problem with its robust counterpart. Many researchers [1, 6, 7, 12] have investigated duality theory for linear or convex programming problems under uncertainty with the worst case approach.

On the other hand, recently, many authors [3, 4, 8, 9, 10, 11, 12] investigated surrogate duality for quasiconvex programming. Surrogate duality is used in not only quasi-convex programming but also integer programming and the knapsack problem [2, 3, 4, 8, 9, 10]. In particualr, Suzuki, Kuroiwa and Lee [12] proved a surrogate duality theorem for an optimization problem involving a quasi-convex objective function and finitely many convex constraint functions with data uncertainty, and a

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similar one for a semi-definite optimization problem involving a quasi-convex objective function and a constraint set defined by a linear matrix inequality with data uncertainty.

In this brief note, we present a surrogate duality theorem for a semi-infinite optimization problem involving a quasi-convex objective function and infinitely many convex constraint functions with data uncertainty.

Consider the following semi-infinite optimization problem in the absence of data uncertainty

$$\begin{array}{lll} \text{(SIP)} & \min & f(x) \\ & \text{s.t.} & g_t(x) \leq 0, \ \forall t \in T \end{array}$$

where $f, g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are functions and T is an infinite index set.

The semi-infinite optimization problem (SIP) in the face of data uncertainty in the constraints can be captured by the problem

$$\begin{array}{lll} \text{(USIP)} & \min & f(x) \\ & \text{s.t.} & g_t(x, v_t) \leq 0, \ \forall t \in T, \end{array}$$

where $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_t(\cdot, v_t)$ is convex for all $t \in T$ and $u_t \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{U}_t \subset \mathbb{R}^q$, $t \in T$.

The uncertainty set-valued mapping $\mathcal{V} : T \to 2^{\mathbb{R}^q}$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. We represent by $v_t \in \mathcal{V}_t$ an element of an uncertainty set $\mathcal{V}_t$ and $v \in \mathcal{V}$ means that v is a selection of $\mathcal{V}$, i.e., $\mathcal{V} : T \to \mathbb{R}^q$ and $v_t \in \mathcal{V}_t$ for all $t \in T$.

The robust counterpart (the worst case) of (USIP):

$$\begin{array}{lll} \text{(RSIP)} & \min & f(x) \\ & \text{s.t.} & g_t(x, v_t) \leq 0, \ \forall v_t \in \mathcal{V}_t, \ \forall t \in T \end{array}$$

We denote by $\mathbb{R}_+^{(T)}$ the set of mapping $\lambda : T \to \mathbb{R}_+$ (also denoted by $(\lambda_t)_{t \in T}$) such that $\lambda_t = 0$ except for finitely many indexes). The robust feasible set F is defined by

$$F := \{x \in \mathbb{R}^n : g_t(x, v_t) \leq 0, \ \forall t \in T, \ \forall v_t \in \mathcal{V}_t\}.$$

The paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we give a surrogate duality theorem for the semi-infinite optimization problem with data uncertainty.

2. Preliminaries

Let $\langle v, x \rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given a set $A \subset \mathbb{R}^n$, we denote the closure of A and the convex hull by A by cl$A$ and co$A$, respectively. The indicator function δ_A is defined by

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{otherwise.} \end{cases}$$

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$.

We denote the domain of f by dom$f$, that is, $\mathrm{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$. The epigraph of f, epi$f$, is defined as $\mathrm{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if epi$f$ is convex. In addition, the Fenchel conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as $f^*(u) = \sup_{x \in \mathrm{dom} f}\{\langle u, x\rangle - f(x)\}$.

Recall that f is said to be quasiconvex if for all $x_1,\ x_2 \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, $f((1-\lambda)x_1 + \lambda x_2) \leq \max\{f(x_1), f(x_2)\}$. Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as $L(f, \diamond, \beta) = \{x \in \mathbb{R}^n \mid f(x) \diamond \beta\}$ for any $\beta \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is a convex set, or equivalently, for any $\beta \in \mathbb{R}$, $L(f, <, \beta)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

Lemma 2.1. [6] *Let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$ (where I is an arbitrary index set), be a proper lower semicontinuous convex function. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\sup_{i \in I} g_i(x_0) < +\infty$. Then*

$$\mathrm{epi}(\sup_{i \in I} g_i)^* = \mathrm{cl}\,(\mathrm{co} \bigcup_{i \in I} \mathrm{epi}\, g_i^*).$$

3. Surrogate duality

In the following theorem, under a constraint qualification, we prove the surrogate duality theorem for the semi-infinite optimization problem with data uncertainty. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an upper semicontinuous quasiconvex function with $\mathrm{dom} f \cap F \neq \emptyset$, and Let g_t be functions from $\mathbb{R}^n \times \mathbb{R}^q$ to $\mathbb{R}$ such that for each $t \in T$ and $v_t \in \mathcal{V}_t$, $g_t(\cdot, v_t)$ is a convex function.

Theorem 3.1. *Assume that the cone,*

$$\bigcup_{(v,\lambda) \in \mathcal{V} \times \mathbb{R}^{(T)}_+} \mathrm{epi}(\sum_{t \in T} \lambda_t g_t(\cdot, v_t))^*$$

is closed and convex. Then we have

$$\inf\{f(x)|g_t(x,v_t)\le 0,\ \forall v_t\in\mathcal{V}_t,\ \forall t\in T\}$$
$$=\max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}}\inf\{f(x)|\sum_{t\in T}\lambda_t g_t(x,v_t)\le 0\}.$$

Proof. Let $m=\inf_{x\in F}f(x)$. Since $F\subset\{f(x)|\sum_{t\in T}\lambda_t g_t(x,v_t)\le 0\}$ for any $(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}$, we have, for any $(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}$,

$$\inf\{f(x)|\sum_{t\in T}\lambda_t g_t(x,v_t)\le 0\}\le m.$$

If $m=-\infty$, then the conclusion holds trivially. So, assume that m is finite.

If $L(f,<,m)$ is empty, then putting $\lambda=0$ and taking any $v\in\mathcal{V}$, $m=\{f(x)|\sum_{t\in T}\lambda_t g_t(x,v_t)\le 0\}$ and hence the conclusion holds.

Suppose that $L(f,<,m)$ is not empty. Then $L(f,<,m)\cap F=\emptyset$, $L(f,<,m)$ is a nonempty open convex set, and F is closed and convex. So, by a separation theorem, there exist a nonzero $x^*\in\mathbb{R}^n$ and $\alpha\in\mathbb{R}$, such that for all $x\in F$ and $y\in L(f,<,m)$,

$$\langle x^*,x\rangle\le\alpha<\langle x^*,y\rangle \tag{3.1}$$

Since $\langle x^*,x\rangle\le\alpha$ for any $x\in F$, $(x^*,\alpha)\in\mathrm{epi}\delta_F^*$. By Lemma 2.1,

$$\mathrm{epi}\delta_F^*=\mathrm{epi}(\sup_{\substack{v\in\mathcal{V}\\ \lambda\in\mathbb{R}_+^{(T)}}}\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^*=\mathrm{cl\,co}(\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}}\mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^*).$$

By assumption,

$$\mathrm{epi}\delta_F^*=\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}}\mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^*.$$

Thus

$$(x^*,\alpha)\in\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}}\mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^*.$$

Hence, there exist $\bar\lambda\in\mathbb{R}_+^{(T)}$ and $\bar v\in\mathcal{V}$ such that

$$(x^*,\alpha)\in\mathrm{epi}(\sum_{t\in T}\bar\lambda_t g_t(\cdot,\bar v_t))^*.$$

So, $(\sum_{t\in T}\bar\lambda_t g_t(\cdot,\bar v_t))^*(x^*)\le\alpha$, that is, $\langle x^*,x\rangle-\sum_{t\in T}\bar\lambda_t g_t(x,\bar v_t)\le\alpha$ for any $x\in\mathbb{R}^n$. Hence, for any $x\in\{x\in\mathbb{R}^n\mid\sum_{t\in T}\bar\lambda_t g_t(x,\bar v_t)\le 0\}$, $\langle x^*,x\rangle\le\alpha$. Thus, from (3.1), for any $x\in\{x\in\mathbb{R}^n|\sum_{t\in T}\bar\lambda_t g_t(x,\bar v_t)\le 0\}$, $x\notin L(f,<,m)$. So, for any $x\in\{x\in\mathbb{R}^n\mid\sum_{t\in T}\lambda_t g_t(x,v_t)\le 0\}$, that is, $\inf\{f(x)|\sum_{t\in T}\bar{\lambda}_t g_t(x,\bar{v}_t)\le 0\}\ge m$. Since $\inf\{f(x)|\sum_{t\in T}\bar{\lambda}_t g_t(x,\bar{v}_t)\le 0\}\le m$, we have

$$\inf\{f(x)\mid\sum_{t\in T}\bar{\lambda}_t g_t(x,\bar{v}_t)\le 0\}=m.$$

So, the conclusion holds. □

Remark 3.2. The assumption in Theorem 3.1 can be called a closed and convex cone constraint qualification. This constraint qualification is a semi-infinite and robust version of the one in [5], and the semi-infinite version of the one in [6].

Corollary 3.3. *Assume that for each $x\in\mathbb{R}^n$ and each $t\in T$, $g_t(x,\cdot)$ is a concave function and there exists $x_0\in\mathbb{R}^n$ such that for all $t\in T$ and all $v_t\in\mathcal{V}_t$, $g_t(x_0,v_t)<0$. Then we have*

$$\begin{aligned}&\inf\{f(x)|g_t(x,v_t)\le 0,\ \forall v_t\in\mathcal{V}_t,\ \forall t\in T\}\\&=\max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}}\inf\{f(x)|\sum_{t\in T}\lambda_t g_t(x,v_t)\le 0\}.\end{aligned}$$

Proof. Following the proof approaches of Proposition 2.3 and Proposition 3.2 in [6], we can check that

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}}\operatorname{epi}\Big(\sum_{t\in T}\lambda_t g_t(\cdot,v_t)\Big)^*$$

is closed and convex. Thus, from Theorem 3.1, the conclusion holds. □

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